The nature of quantities influences the representation of arithmetic problems: evidence from drawings and solving procedures in children and adults

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Abstract

When solving arithmetic problems, semantic factors influence the representations built (Gamo, Sander & Richard, 2010). In order to specify such interpretative processes, we created structurally isomorphic word problems that could be solved with two distinct algorithms. We tested whether a distinction between cardinal and ordinal quantities would lead solvers, due to their daily-life knowledge, to build different representations, influencing their strategies as well as the nature of their drawings. We compared 5th grade children and adults in order to assess the validity of this hypothesis with participants of varying arithmetic proficiency. Results confirmed that the distinction between cardinal and ordinal situations led to different solving strategies and to different drawings among both age groups. This study supports the cardinal versus ordinal ontological distinction and calls for the integration of the role of daily-life semantics when accounting for arithmetic problem solving processes.

Keywords: arithmetic problem solving; interpreted structure; semantic encoding; strategy choice.

Introduction

What are the steps between reading an arithmetic word problem and implementing a set of mathematical operations, and how can they be studied? It is well established, since Riley, Greeno and Heller (1983) proposed their typology of additive word problems, that different problem statements lead to different performances. Yet, the reasoning processes and representations accounting for such differences remains controversial.

The schema theory (Kintsch & Greeno, 1985) proposes that solving a word problem requires to select and to instantiate a schema fitting the problem at hand. For example, any comparison problem will require to retrieve the corresponding schema and to implement it with the available values (Riley et al., 1983). However, it has been argued that this approach underestimates interpretative effects. For instance, Hudson (1983) showed that young children had much more trouble solving a problem stating "there are 5 birds and 3 worms, how many more birds is there than worms?" than they had solving a problem in which the question was "how many birds won't get a worm?". This result shows that two problems sharing the same schema can lead to different performances.

A contrasting approach comes from Johnson-Laird's (1983) theory of mental models. It posits that, during reading, a mental representation is constructed in working memory, the structure of which is analogous to the structure of the situation depicted in the problem statement (Reusser, 1990). This representation depicts the meaningful relations between the elements of the problem. The idea of a problem-specific representation integrating conceptual information from the problem statement can account for interpretative effects described in the literature. De Corte, Verschaffel and De Win (1985) showed that rewording a problem statement so that the semantic relations are made more salient facilitates the solving process. Similarly, introducing daily-life situations in the cover stories of word problems contributes to better performance (Stern & Lehrndorfer, 1992; Vlahović-Štetić, 1999). The use of specific words or sentences can modify the representation constructed by the solvers (Cummins, Kintsch, Reusser & Weimer, 1988). In a study challenging the predictions of mental model and schema views, Thevenot, Devidal, Barrouillet and Fayol (2007) showed that placing the question at the beginning instead of at the end of a problem statement provided more benefit to the less experienced solvers. This result supported the mental model theory, whereas a schema account would have predicted the reverse pattern.

Semantic determinants of problem solving

The issue of the semantic determinants of the solvers' mental representations is of importance. Bassok, Wu and Olseth (1995) showed that the semantic relations connecting a problem's entities influence analogical transfer. They contrasted problems where objects were given to people (OP) and problems were people were assigned to objects (PO). They found that, since in real life objects are usually given to people rather than people being assigned to objects, OP training examples led to better performance with OP transfer problems than with PO transfer problems.

Along this line, Bassok, Chase and Martin (1998) asked participants to create addition or division problems involving specific sets of objects that were provided. They showed that when the objects shared a functionally asymmetric semantic relation (e.g. apples and baskets evoke the *contain* relation), participants tended to create division problems, whereas they created addition problems when using functionally symmetric sets of objects (such as oranges and apples, that belong to the same superordinate fruit category). These biases are not driven by arithmetic properties but rather by the world semantics. Bassok (2001) developed the *semantic alignment* framework proposing that the solvers abstract an *interpreted structure* that depends on their world knowledge about the entities described in the problem statement. This interpreted structure integrates the structural role of the entities mentioned in the problem, and can thus lead to an appropriate use of abstract formal knowledge when the relations it describes are semantically aligned with the mathematical relations of the problem (Bassok et al., 1998; Bassok, 2001). Both behavioral (Bassok, Pedigo & Oskarsson, 2008) and physiological (Guthormsen et al., 2015) measures confirmed that problem solving is easier when daily-life knowledge (world semantics) and knowledge about mathematical concepts (mathematical semantics) are aligned with each other.

Investigating participants' representations

The semantic alignment framework predicts that representations abstracted from problem statements influence solver's solving strategies. Yet, the key semantic dimensions influencing the representations and explaining the lack of transfer remain to be elucidated in order to promote methods to help students overcome the incompatibilities posed by a problem.

In this regard, problems with multiple solving strategies are of particular interest to study representations, since the selection of one strategy over another is informative about the representation constructed by the solvers (De Corte, Verschaffel & De Win, 1985). For instance, Thevenot and Oakhill (2005) worked on a multiple-step problem solving task in which the cognitive load was manipulated through values size (large or small). They showed that depending on the size of the values, participants used different solving algorithms. The issue of the semantic determinants of problem representations can be tackled using such a paradigm in which different solving strategies are available, and the solver's ability to pick and use one tells us about the abstracted interpreted structures (Hakem, Sander, Labat & Richard, 2005). For example, Coquin-Viennot & Moreau (2003) showed that the presence of a grouping element in a problem statement (such as flowers presented within a bouquet instead of separately) could incite participants to use a factorizing rather than a development algorithm.

Another way to study the participants' mental representations is the use of drawings. Vosniadou and Brewer (1992) elicited drawings from 3rd and 5th grade children so as to study the development of their representations of the earth. As for problem solving, studies have highlighted the link between problem representations and drawings of the situations (Barrios & Martinez, 2014; Edens & Potter, 2007). Drawings are thus an accurate way to gather information regarding the solvers representations.

Encoding ordinal and cardinal quantities

In the following experiment, we will capitalize on problems that can be solved with two different strategies. Previous studies have suggested that one can draw an ontological distinction between two types of situations involving numerical values: cardinal situations, consisting of sets of unordered elements, and ordinal situations, where units are endogenously ordered and can be represented along an axis, such as a timeline (Gamo et al., 2010; Hakem et al., 2005; Sander & Richard, 2005). When solving an arithmetic word problem, the authors posited that solvers abstract an interpreted structure that is aligned with either a cardinal or an ordinal representation.

These two types of representations elicit different solving strategies: in ordinal representations, subtractions are seen as calculations performed on a one-dimensional ordered scale, whereas in cardinal representations they are encoded as a difference between a whole and a component part (Hakem et al., 2005). Thus, according to Gamo et al.'s hypothesis, a subtraction could either be perceived as a comparison or as a complementation, depending on the situation described in the problem statement. The paradigm developed by Hakem et al. consisted in problems that admitted two distinct solving strategies that were implementable for both cardinal (number of people) and ordinal (duration) quantities. Problem statements 1 and 2 below embody this distinction between cardinal problems and ordinal problems:

- Problem 1: "There are 5 people in the Richard family. When the Richards go on holidays with the Roberts, they make a total of 14 people at the hotel. The Roberts are joined on holiday by the Dumas family. In the Dumas family, there are 3 people less than in the Richard family. The Roberts are going on holidays with the Dumas. How many will they be at the hotel?"
- Problem 2: "Antoine took painting classes for 5 years, and stopped at the age of 14. Jean started at the same age as Antoine, and went to classes 3 years less than him. How old was Jean when he stopped attending painting classes?"

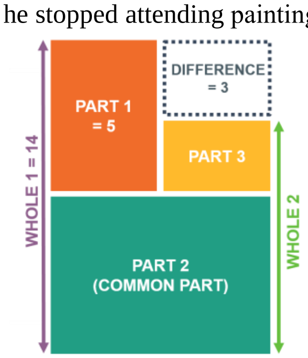

Figure 1: Structure of the problems. This structure can depict both problems and is compatible with both strategies.

Problems 1 and 2 are isomorphs sharing the same deep structure (Figure 1), and can both be solved using either of two strategies: either a 3-step complementation strategy (14

– 5 = 9; 5 – 3 = 2; 9 + 2 = 11) or a 1-step matching strategy (14 – 3 = 11). Yet, because the quantities used are different, the authors hypothesized that (i) the interpreted structures abstracted are too, each problem statement consequently favoring the use of one strategy over the other; and that (ii) the unequal distribution of strategies used may be accounted for by the nature of the representations abstracted: problem 1 encoded as a cardinal problem (Figure 2) and problem 2 encoded as an ordinal problem (Figure 3).

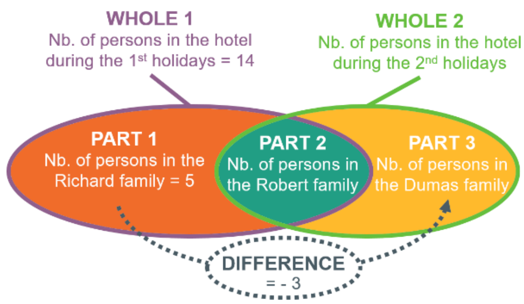

Figure 2: Cardinal representation of problem 1. This interpreted structure fosters the calculation of the intersection (part 2) between whole 1 and whole 2, thus favoring the 3-step complementation strategy.

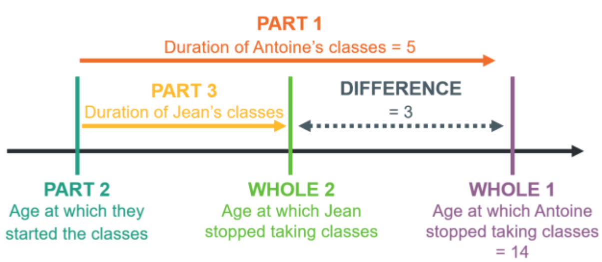

Figure 3: Ordinal representation of problem 2. This interpreted structure puts forward the fact that the difference between whole 1 and whole 2 is equal to the difference between part 1 and part 3. The shorter 1-step comparison algorithm thus becomes available to solve the problem.

In accordance with the authors' hypothesis, the participants who were asked to solve the problems using as few operations as possible found the 1-step matching strategy on problem 1 in less than 5% of the cases. On the other hand, problem 2 led to a rate of use of the 1-step matching strategy over 60%, suggesting that comparisons are indeed made salient in ordinal representations. Hakem et al.'s (2005) study of the solving strategies showed that the two types of problems were underlain by different representations. Yet, the claim that ordinal and cardinal quantities evoke the corresponding ordinal and cardinal representations needs further empirical support.

Present study

Our study builds on the work of Hakem et al. (2005) in order to highlight the role of the general semantic features on the representations abstracted by the solvers and on the implemented solving strategies. We aimed at providing converging measures of the impact of the cardinal/ordinal distinction on the solvers' ability to solve the problems, and to provide the first empirical test of these effects on children and adults simultaneously. To this end, 5th graders as well as adults were asked to perform two tasks: solving problems involving different types of cardinal and ordinal variables using as few operations as possible, and making a drawing for each problem.

The goal of the experiment was twofold: first, we intended to confirm with both age groups the validity of the ordinal versus cardinal distinction with a new material including new types of quantities and using more systematically controlled problem statements. This was intended to show that strong semantic effects affect both younger and older – more proficient – participants in arithmetic problem solving. Second, we wanted to show that those effects originate in the representations abstracted from the problems, and translate into the algorithms implemented by the solvers. We predicted that within each group, the mean percentage of the 1-step matching strategy would be significantly lower on cardinal problems than on ordinal problems, despite the adults achieving a globally higher solving performance than the children. Also, we predicted that for each age group the drawings would reveal a higher ordinal versus cardinal ratio of distinctive features for ordinal than for cardinal problems.

Experiment

Participants

We recruited samples from two populations for this study: a group of 59 children in 5th grade (27 females, M = 11.00 years, SD = 0.36), and a group of 52 adults (36 females, M = 26.86 years, SD = 9.72). All participants were recruited from the Paris region and spoke French fluently. None had previously participated in any similar experiment.

Materials and procedure

Each participant was presented with a set of 12 problems, 6 using ordinal values (duration, height or number of floors), and 6 using cardinal values (number of elements, price or weight), according to Hakem et al.'s definition. We considered duration, height and number of floors as ordinal values because their ordinal component is salient in daily life, putting emphasis on successorship relation and on comparison. Similarly, number of elements, price and weight were used as cardinal values because the world semantics attached to such quantities evoke the unordered grouping of elements assigned to values and the partition of a whole into its component parts.

All the problems had the same number of sentences. The numerical values were provided in the same order and both numerical values and problem order were randomized between participants. The problems were printed on 13-page booklets with the instructions detailed on the first page. The participants were asked (1) to solve the problems using as few arithmetic operations as possible, (2) to write down every operation they made, even those they solved using mental calculation, and (3) to make a drawing for each problem statement that could help someone else understand and solve the problem. Each page was divided into four parts: problem statement, 'draft' area, 'response' area and 'drawing' area. The booklets and instructions were strictly identical for both age groups.

Coding

The successful strategies were categorized either as correct 1-step matching strategy, or as correct 3-step complementation strategy. A problem was considered correct when the expected result was obtained and accompanied by calculations[1]. Regarding the drawings, we designed an 8-item rating scale evaluating to what extent the drawings possessed ordinal versus cardinal characteristics. The scale included 4 cardinal items (Figure 4.a) and 4 ordinal items (Figure 4.b). The items were chosen so that they would either depict unordered elements being grouped in sets and embedded sets, or ordered elements being described as positions on an axis and compared along this axis.

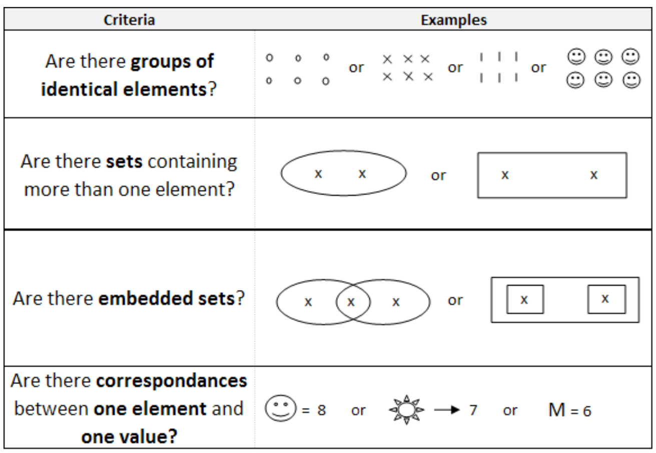

Figure 4.a: Coding grid for cardinal features.

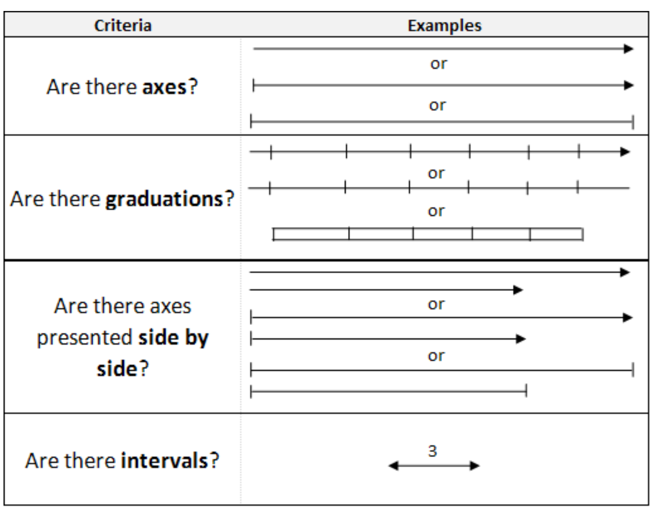

Figure 4.b: Coding grid for ordinal features.

[1] When a calculation error resulted in a difference of +1 or −1 compared to the correct value, problems were still considered correctly solved.

Each drawing, including those associated with failed problems, was scored by two independent raters who were not familiar with the theories at play and ignored the hypotheses being tested. They were asked to rate the drawings according to the 8 items scale resulting from the aggregation of Figures 4.a and 4.b. After an initial rating phase, the percent agreement between the two raters was of 91.87%. An inter-rater reliability analysis using Cohen's Kappa statistic showed substantial agreement between raters. ($\kappa = 0.61$, $SE = 0.14$), according to Landis & Koch's typology (1977). After discussion, the raters reached 100% agreement. For each drawing, a score was then calculated by subtracting the number of cardinal items to the number of ordinal items, thus creating a scale ranging from −4 (the most cardinal) to +4 (the most ordinal).

Results

Our first hypothesis was that problems with ordinal quantities should facilitate the use of the matching 1-step strategy compared to problems with cardinal quantities. In both groups, we evaluated whether participants did use the 1-step matching strategy more often on problems involving ordinal world semantics as compared to problems with cardinal world semantics, as hypothesized.

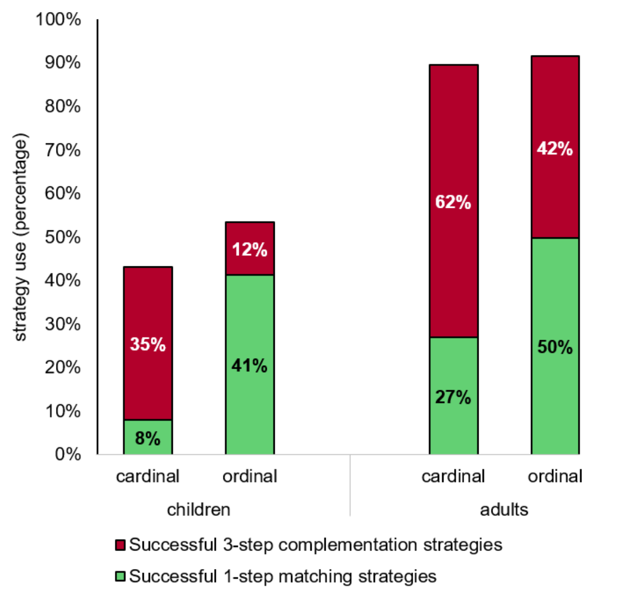

Figure 5: Children's and adults' mean rate of use of the two solving algorithms depending on the type of quantity used.

Figure 5 details the participants' use of each strategy depending on the type of quantity featured in the problems. A paired t-test analysis revealed that the mean rate of use of the 1-step matching strategy was higher on ordinal (M=0.39, SD=0.31) than on cardinal (M=0.08, SD=0.17) problems ($t(58)$=8,36 $p < 0.001$). The same analyses were performed for the adults and showed that the mean rate of use of the 1-

step matching strategy was also higher on ordinal (M=0.457, *SD*=0.33) than on cardinal (M=0.253, *SD*=0.35) problems (t(51)=4.99, p<0.001). This confirmed that the choice of a solving algorithm is influenced by the cardinal versus ordinal nature of the quantities and that this effect is robust among adults. Additionally, the 1-step algorithm was significantly less used by children than by adults on cardinal (t(109)=3.48, p < 0.001, unpaired t-test) but not ordinal (t(109)=1.10, *p* = 0.27, unpaired t-test) problems, meaning that children had significantly more difficulty than adults using the 1-step strategy on cardinal, but not on ordinal problems.

To test our second hypothesis, we focused on the drawings made by the participants. Figure 6 details the rating of the drawings depending on the type of quantity used in the problems. Drawing score was significantly lower for drawings depicting problems with cardinal quantities (M=–0.55, *SD*=0.78) than for those describing problems with ordinal quantities ((M=0.06, SD=0.87), (t(58)=5.61, p < 0.001, paired t-test), indicating that problems using ordinal quantities led young participants to draw ordinal features (axes, intervals, etc.) at a higher ratio over cardinal features (sets, groups of elements, etc.) compared to the ordinal problems.

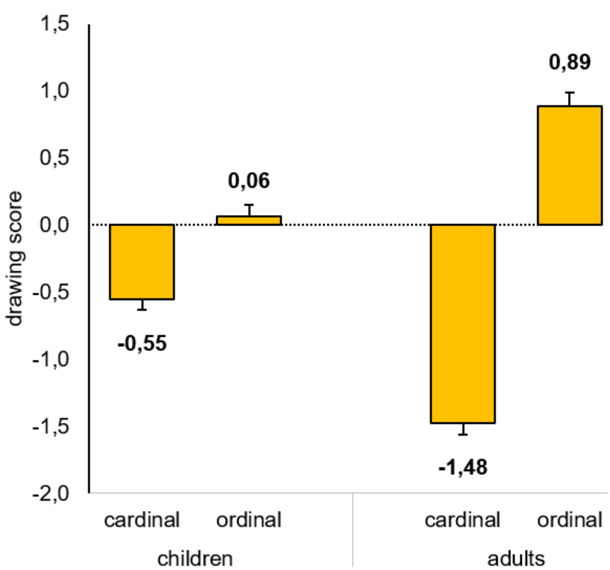

Figure 6: Children's and adults' mean drawing score depending on the type of quantity used in the problems. Vertical bars denote 0.95 confidence intervals.

Similarly, among adult participants, problems with cardinal quantities (M=–1.48, *SD*=0.79) led to a significantly lower drawing score than problems with ordinal quantities (M=0.89, SD=0.86), (t(51)=12.44, p < 0.001, paired t-test). In sum, the presence of ordinal (resp. cardinal) quantities seems to result in representations featuring a higher number of ordinal (resp. cardinal) features, in both children and adults. Of note, drawing score was significantly higher among children than among adults on cardinal problems (t(109)=6.24, p < 0.001, unpaired t-test) whilst significantly lower among children than among adults on ordinal problems (t(109)=5.00, p < 0.001, unpaired t-test); in other words, children included significantly less cardinal features than adults while representing cardinal problems, and significantly less ordinal features than adults when representing ordinal problems.

Discussion

The fact that the cardinal versus ordinal distinction in problem statements influenced both children's and adults' solving strategies confirmed the robustness of these interpretative effects, even with experienced solvers who should not meet any difficulty in solving such simple problems. Indeed, children performed about half as well as adults, yet the distinction between cardinal and ordinal problems remains significant in both populations. Additionally, adults' performances were significantly higher on cardinal, but not on ordinal problems, indicating that when semantically congruent with the 1-step strategy, world semantics help children achieve adult-like performance on the task.

The elicited drawings provided an empirical confirmation of the importance of the ordinal versus cardinal distinction in both populations. The fact that children produced drawings that had significantly fewer ordinal (resp. cardinal) features on ordinal (resp. cardinal) problems may be attributed to a global lack of details in their drawings, that nevertheless did not prevent a significant distinction between cardinal and ordinal drawings to appear among children. Additionally, children may have more difficulties to produce a graphic implementation of ordinal situations, which would explain their poor ordinal score (0.06) for ordinal problems. This calls for further research on the topic.

Overall, the results of both the drawing and the solving tasks show that the participants' semantic knowledge about the quantities used in the problems (i.e. their experience with counting the number of apples in a bag, adding the price of every item on a bucket list, calculating the arrival time of their train or using the elevator) have a decisive influence on the encoding of arithmetic word problems. The distinction introduced between ordinal and cardinal problem statements was reflected in the representations constructed (as shown by the drawings made by the participants) and led them to use different solving algorithms, even when asked specifically to use the shortest strategy they could find. Furthermore, the fact that those effects could be highlighted both with young pupils and adults indicates the robustness of such encoding constraints. The ontological distinction hypothesized between ordinal and cardinal representations was thus confirmed on two complementary tasks.

The use of a double measure of the influence of the solvers' knowledge about the world allowed us to gather converging clues shedding light both into the representations abstracted and into the algorithms subsequently implemented. By focusing on the role of semantic properties on the initial encoding of a problem, we hope to gain a finer understanding of arithmetic problem solving in its whole, and to pave the way for accounting for the interactions between world semantics, mathematical semantics and algorithms.

Understanding the determinants of problems' representations is a crucial step to identify the potential pitfalls and dead ends born from semantic incongruence (Gros, Sander & Thibaut, 2016) as well as to help develop analogical transfer between isomorphic problems by promoting semantic recoding among the learners (Gamo, Sander & Richard, 2010; Gros, Thibaut & Sander, 2015). Doubtlessly, the educational opportunities resulting from a better understanding of the conditions necessary for semantic recoding and analogical transfer between problems are promising.

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