"This problem has no solution": when closing one of two doors results in failure to access any.

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Abstract

We investigated what happens when the spontaneous encoding of a problem is incongruent with its solving strategy. We created word problems from which two distinct semantic representations could be abstracted. Only one of these representations was consistent with the solving strategy. We tested whether participants could recode a semantically incongruent representation in order to access another, less salient, solving strategy. In experiment 1, participants had to solve arithmetic problems and to indicate which problems were unsolvable. In experiment 2, participants received solved problems and had to decide whether the solution was appropriate or not. In both experiments, participants had more difficulties acknowledging that problems inducing an incongruent representation could be solved than they had for problems inducing a congruent representation. This was confirmed by response times. These results highlight how semantic aspects can lead even adults to fail or succeed in the solving of arithmetic problems requiring basic mathematical knowledge.

Keywords: arithmetic problem solving; semantic structures; semantic encoding; strategy choice.

Introduction

The view that solvers build an abstract problem schema that guides a solving strategy free of contextual effects such as the semantic content of the problem has been challenged by recent investigators (e.g. Kotovsky & Simon, 1985; Bassok & Olseth, 1995). They have argued that the representation of the situation (objects, relations between objects) of a problem influences the solving procedure by reference to mental models (Reusser, 1990) that are mental constructs analogous to the situation depicted in the problem. Further work has shown that elements that are irrelevant from a mathematical point view (e.g. the type of objects referred to in the problems) were taken as cues while constructing an interpreted structure that solvers were using as a frame during the solving process (Bassok, Wu & Olseth, 1995). Recent studies have highlighted the importance of such a framework (Bassok, DeWolf & Holyoak, 2015; Lee, DeWolf, Bassok, Holyoak, 2016). The existence of an abstract interpreted structure raises the question of its congruence with the algorithm relevant for the solution. For example it has been shown that spontaneously, additions are associated with objects belonging to the same level of categorization, such as oranges and apples, and not to functionally connected objects such as oranges and baskets, even though there is no mathematical reason why one would not add these two types of objects (Bassok, 2001). To what extent can the algorithm be used when the interpreted structure is not congruent with the mathematical one? To date, it remains an open issue to show whether an incongruence between the representation abstracted from the wording and the mathematical structure will lead the solver to a dead end, even if the solving algorithm is one which only requires mathematical skills already mastered by the solver.

In order to investigate the representational issues, recent studies focused on problems that were solvable by alternative strategies, each one relying on a specific representation (Thevenot & Oakhill, 2005, Coquin-Viennot & Moreau, 2003, Gros, Thibaut & Sander, 2015). They showed that the strategy that was used to solve the problem was strongly constrained by the representation solvers were relying on. The present study is based on this type of problems. The main question addressed regards what participants do when the solving strategy associated with the representation spontaneously evoked by the problem is prohibited due to the lack of some necessary information. In other words, what happens when the "door" the participants usually use is blocked? Will solvers switch to an alternative representation congruent with the other strategy, which is still practicable, or will they remain stuck to their initial representation and think that the problem has no solution? In the latter case, it would mean that their initial representation of the problem is so constraining that they became blind to the alternative one and to the strategy associated with it. The solvers would thus be incapable of noticing and using the other "door", yet wide open and leading to the solution.

Content effects

The characteristics of the situation depicted in a word problem have been shown to influence the solver's ability to find the correct solution of a problem. Isomorphic problems sharing the same mathematical structure can thus have different levels of difficulty depending on their wording (De Corte, Verschaffel, De Win, 1985).

More specifically, the influence of a wide range of abstract semantic properties on the encoding of word problems has been highlighted in different studies: continuous versus discrete property of a constant change (Bassok & Olseth, 1995), continuous versus discrete property of the entities of a problem (DeWolf, Bassok & Holyoak, 2015), symmetrical versus asymmetrical property of a relation (Bassok, Wu & Olseth, 1995), categorical versus functional property of an association between elements (Bassok, Chase & Martin, 1998) have all been shown to influence the solving strategies used by the solvers.

Overall, the crucial role of such abstract semantic features on the representations involved in problem solving demonstrates that any solving task necessarily depends on the solver knowledge regarding the elements included in the problem. This world knowledge can ease the solving process when the semantic information is congruent with it, or it can inversely impair the solving of a problem if the available information leads to a representation that is not congruent with the solving strategy.

Multiple-strategy word problems

In order to address this issue, Gamo, Sander & Richard (2010) studied multiple-strategy word problems in which the solution could be found in two different ways: either a 3-steps strategy requiring to perform a complementation inference (where the value of the difference between a set and a whole is calculated), or a 1-step strategy based on a comparison inference (requiring the calculation of the difference between two homologous quantities). According to Bosc-Miné and Sander (2007), it is easier to make a complementation inference when the context highlights the part/whole relation between the elements, as is the case in problems featuring cardinal values. On the other hand, an ordinal representation of the values usually highlights the comparison relations between the featured values, and thus promotes comparison inferences. This effect is fairly intuitive when we consider prototypes of these two categories of problems:

- Problem A: “John bought a 5€ exercise book and scissors. He paid 14€. A pen costs 3€ less than the exercise book. Paul bought scissors and a pen. How much did he pay?”
- Problem B: “Antoine took painting classes for 5 years, and stopped when he was 14 years old. Jean began at the same age as Antoine and took the course during 3 fewer years. At what age did Jean stop?”

Both types of problems can be solved using two identical strategies, namely the complementation-difference strategy (14-5=9 then 5-3=2 then 9+2=11, in which 11 is the answer) and the comparison-difference strategy (14-3=11, in which 11 is the answer). Yet, data collected by Gamo et al. (2010) showed that the comparison strategy is only scarcely used (4% of the situations) on problems similar to the problem A, which evoke a cardinal part/whole representation, whilst this strategy is widely used (64% of the problems) to solve the problems comparable to the problem B, in which the difference between the duration of the two classes is directly encoded in the ordinal representation of the problems. On the other hand, using the comparison on problem A requires to identify that the difference between the total amount paid by Paul and by John is equal to the price difference between the pen and the exercise book. The price of the scissors doesn’t need to be calculated to solve the problem.

In order to gain a better understanding of these two types of induced representations, it is possible to draw a schema of their structure. Regarding the problem A, Gamo et al. (2010) suggested that the interpreted representation is cardinal. We can thus represent the different elements of the problem as depicted in figure 1.a.

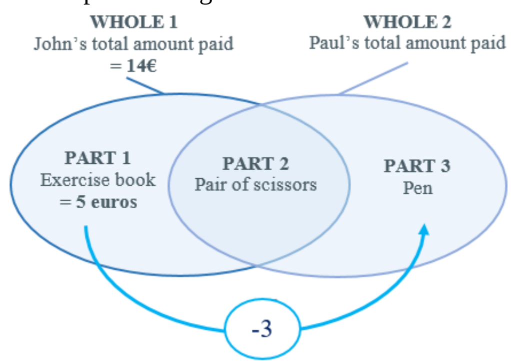

Figure 1.a: Cardinal representation of headcount problems such as those found in Gamo, Sander & Richard (2010).

It is then clear that the representation promotes the calculation of the intersection (Part 2) between Whole1 and Whole2 in order to get the value of Whole2, thus favoring the complementation strategy. The corresponding calculations are as follows: *Whole1 – Part1 = Part2*, then *Part1 – Difference = Part3*, then *Part2 + Part3 = Whole2*.

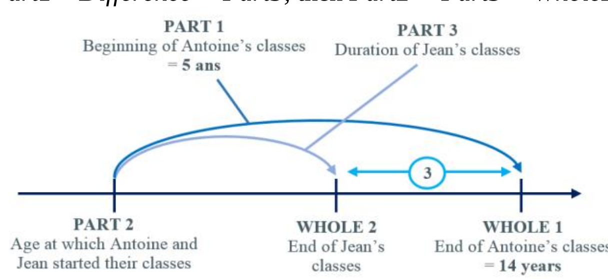

Figure 1.b: Ordinal representation of age problems such as those found in Gamo, Sander & Richard (2010).

Now regarding problem B, the expected representation has an ordinal nature, and can be drawn as on Figure 1.b. The ordinal representation of problem B values puts forward the fact that the difference between Whole1 and Whole2 is equal to the difference between Part1 and Part3. It is straightforward to deduct that the quickest solving strategy is the 1-step comparison strategy: *Whole1 – Difference = Whole2*.

The solving strategy thus is directly encoded in the representation built, despite the fact that these two problems are isomorphic. Only an expert able to have a more exhaustive comprehension of the mathematical relations between the elements of these problems could access an abstract representation subsuming the other two, as detailed in Figure 1.c.

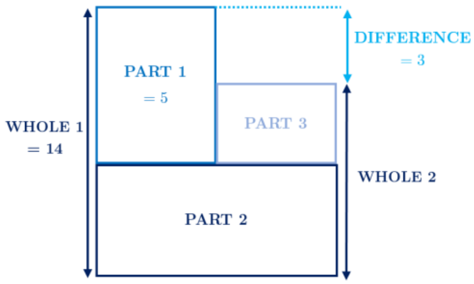

Figure 1.c: Abstract representation of the mathematical structure of the problems found in Gamo et al. (2010).

This structure can equally depict both problems, and makes it possible to use both solving strategies. The variations observed in the choice of one strategy over the other clearly indicate that the participants only abstracted intermediate, context-specific representations (with either ordinal or cardinal structure) instead of the abstract mathematical structure of the problems. It therefore seems that the semantic nature of the quantities used has a major impact on the type of representation being evoked, and subsequently on the type of solving strategy being used.

In what follows, we used the two types of problems described above. The above analysis shows that the quantities grounding each solving strategy differ. We gave participants problems in which one quantity was missing. As a result, they could not be solved according to one strategy (the 3-steps strategy) while remaining solvable according to the other. The issue was whether participants would use the other solving strategy (the 1-step strategy) in all cases or, by contrast, whether they would more often fail to solve them when their spontaneous strategy depended on knowing the value of the missing data than when the missing data was not necessary for making use of their spontaneous solving strategy.

Experiment 1

Materials

We created a set of problems which shared the same mathematical structure as described in figure 1.c, with one notable exception: we removed one of the numerical values so that participants could not access the 3-steps complementation strategy anymore, and solely had to rely on the 1-step matching strategy in order to solve the problems. Instead of asking participants to solve the problems with the fewest possible steps, we tested whether the preference for the semantically congruent strategy could lead to failure to solve the problems when only one of the two strategies was available.

We thus created two types of problems: (i) *solvable problems*, in which we removed the "part 1" value, making it impossible to use the 3-steps complementation strategy (see left part of figure 2), and *unsolvable problems*, where the value of Whole1 was not given, making it impossible to solve the problems using any strategy (see right part of figure 2). The latter were used as distractors.

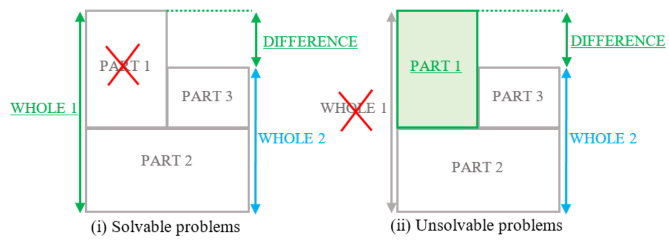

Figure 2: Structure of both solvable and unsolvable problems. Underlined values are those given in the problem. The goal was in both cases to find the value of Whole 2.

We implemented those two types of problems with different quantities inducing either an ordinal representation or a cardinal representation. Six different quantities were used in total; three ordinal quantities (duration, height, and number of floors) and three cardinal quantities (weight, price, number of elements). For each quantity, we created two different problems so as to make sure that the effects measured could not be attributed to one specific context. Those 12 problems were present in both forms: solvable problems and unsolvable problems, making a total of 24 problems in the pool. Finally, the problems were all written in the same way: the same number of sentences was used, and the numerical values (each below 15 so as to limit the calculus induced difficulties) were presented in the same order.

Procedure and experimental design

Each participant was presented with 12 different problems: 2 for each quantity, a solvable problem and an unsolvable one. The 6 unsolvable problems acted as distractors since we tested our hypotheses with the solvable problems. The order in which the problems were presented was randomized for each participant, and so were the two different versions of each problem.

The experiment took place online, on the survey platform Qualtrics. Participants who agreed to participate in the experiment were asked to complete the entire survey. On the first page, the following instructions were given: "You will find an arithmetic problem on each page of this survey. Please take the time to read and understand each of these problems, as this is not a speed test. Your task is to identify which problems can be solved and to indicate for each of them the operation you used to solve them as well as the solution you found. Be careful: some of the problems cannot be solved with the available information."

On each page of the survey, a problem was displayed with the following question below it "Given the data provided, is it possible to find the solution?" with two buttons "Yes" and "No". When the participants pressed "Yes", two new questions appeared, asking them to indicate both the operation needed to solve the problem and the result of the operation. Participants used the keyboard to write down their answers. After participants answered all 12 problems, a new page was displayed asking them for their gender, date of birth, and whether they took any break during the completion of the experiment.

Participants

A total of 89 adults completed our experiment. We removed from the analyses 15 participants who either mentioned taking a break during the test or answered at least one of the questions in less than 5 seconds or more than 5 minutes, which gave 74 participants (44 females, M=33.8 years, SD=13.4 years). Subjects were all fluent French speakers and were recruited through social networks and emails.

Hypotheses

Our first hypothesis regarded the ability to correctly solve the problems that induced a representation which was semantically incongruent with the solving strategy. We hypothesized that participants would have a better rate of success on solvable problems with ordinal quantities compared to solvable problems with cardinal quantities, because participants whose first spontaneous representation was associated with a 3-step strategy would fail to switch to a representation of the problem associated with the 1-step strategy.

Our second hypothesis regarded the situations in which participants overcame the difficulty related to an incongruent representation and managed to find the correct solution of a cardinal problem. We hypothesized that higher response times would be recorded on successfully solved cardinal problems compared to successfully solved ordinal problems.

Results

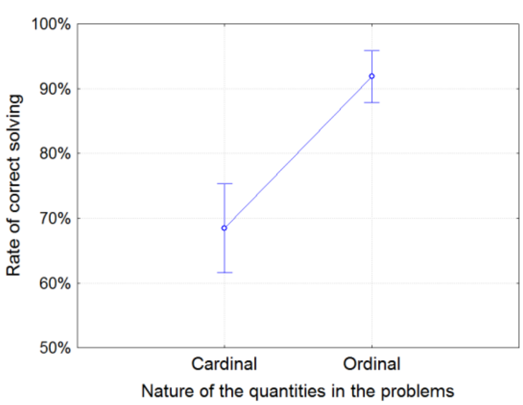

Figure 3: Rate of correct solving on solvable problems. Vertical bars denote 0.95 confidence intervals.

In order to test our first hypothesis, we analyzed the ratio of correct answers on solvable problems, depending on the cardinal or ordinal nature of the quantities used. Ordinal solvable problems were successfully solved in 91.9% of the trials, and cardinal solvable problems in only 68.5% of the trials (see Figure 3). A paired t-test was performed on participants' mean rate of success for cardinal and ordinal problems and showed that the difference was statistically significant ($t(73)=6.38$, $p<0.001$), therefore confirming our first hypothesis.

We then studied the response times on correctly solved problems, depending on the nature of the quantities in the problems. We wanted to see whether accessing the correct 1-step matching strategy on problems inducing a semantically incongruent cardinal representation would require a higher time than needed for problems inducing an ordinal representation. We removed from the analysis 4 participants who did not manage to correctly solve at least 1 cardinal problem and 1 ordinal problem. On average, participants took 68.7 seconds to successfully solve cardinal problems, against 49.8 seconds for ordinal problems (see Figure 4). A linear mixed model applied to the response times of successfully solved problems confirmed that the difference was significant ($F(1,7369)=20.38$, $p<0.001$), thus verifying our second hypothesis.

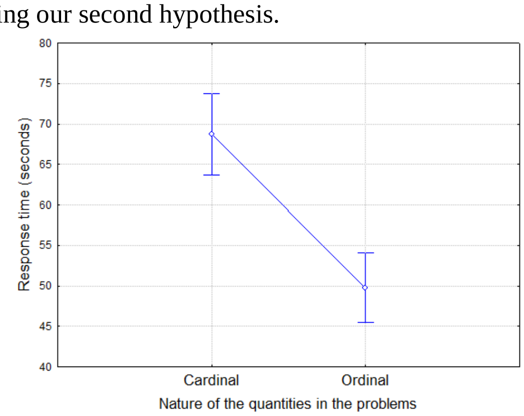

Figure 4: Mean response time on correctly solved problems. Vertical bars denote 0.95 confidence intervals.

Discussion

The results obtained confirmed that depending on the semantic nature of the quantities present in arithmetic word problems, participants induced a semantic representation which was either semantically congruent (ordinal) or semantically incongruent (cardinal) to the only solving strategy available. The fact that the rate of success was significantly different between these two types of problems showed that the semantic congruence does not just promote one strategy over the other (see also Gamo, Sander & Richard, 2009; Gamo, Taabane & Sander, 2011), but can actually lead participants to failure when only one strategy is available. Therefore, a semantically incongruent representation can significantly impair the solver's ability to find the solution of a problem if no semantic recoding takes place.

The study of the response times gave us insights into the need of such a recoding process. The longer time needed to solve cardinal problems suggested the existence of an extra reasoning step which allowed the solver to access a semantically incongruent solving strategy. The fact that the response rates were different for cardinal and ordinal problems suggested that such a process does not systematically happen and the longer RTs showed that this recoding step was a costly one.

In order to better understand these processes, in a second experiment, we explicitly gave the answer to the participants, and asked them to evaluate its validity. The goal was to go beyond the first experiment by evaluating whether the difficulty lied in the ability to discover the solution, or in the capacity to recognize it as a valid one. We made the hypothesis that, even when presented with the complete solution, participants would tend to refuse it more often if the representation they inferred from the problem was semantically incongruent with the solution. They would essentially refuse to use the door presented in front of them, because of their semantically biased representation of the situation.

Experiment 2

Procedure

The only aspect in which this experiment differed from the previous one was the nature of the task proposed to the participants. Instead of telling them to solve the problems themselves, we proposed a potential solution for each problem, and asked them to identify whether the solution was a valid one, or whether the problem could not be solved. For every problem, the question "Given the data provided, is it possible to find the solution?" was displayed, and below two choices were available: "No, we do not have enough information to solve this problem." and "Yes: *numerical value 1 – numerical value 2 = answer. Sentence presenting the answer*". For example, on one of the "number of floors" problems, the following solution was given: "Yes: 11-2=9. Karine arrives at the 9th floor."

Participants

A total of 223 adults completed our experiment. 27 of them were removed from the analysis according to the same criteria as in the previous experiment, and the analyses were performed on the remaining 196 participants (135 females, M=34.5 years, SD=14.8 years). Subjects were native French speakers recruited through social networks and emails.

Hypotheses

We expected to find the same results as in the previous experiment. Even though the solution was directly given to the participants, we hypothesized that they would still refuse it more often for problems with cardinal quantities than for ordinal problems. Similarly, we hypothesized that the correct identification of a solution would require more time for cardinal problems than for ordinal ones.

Results

As in the previous experiment, we first analyzed the ratio of correct answers on solvable problems depending on the semantic nature of the quantities used. The results detailed in Figure 5 showed that the solvable cardinal problems did indeed lead to a lower rate of success (63.6%) than the solvable ordinal problems (88.4%).

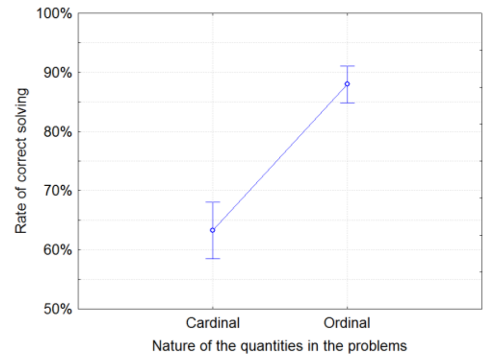

Figure 5: Rate of correct answers on solvable problems. Vertical bars denote 0.95 confidence intervals.

A paired t-test performed on the participants' mean rate of success confirmed that this difference was significant (t(195)=9.26, p<0.001).

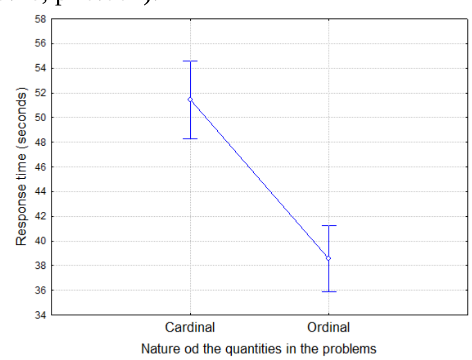

Figure 6: Mean response time on problems where the solution was correctly identified. Vertical bars denote 0.95 confidence intervals.

In order to assess the validity of our second hypothesis, we then studied the response time associated with the correct answers on solvable problems, thanks to a linear mixed model. We removed from the analysis the 26 participants who did not manage to correctly solve at least 1 cardinal problem and 1 ordinal problem. Figure 6 shows that getting a correct answer required a shorter reasoning time for ordinal (38.6 seconds) than for cardinal problems (51.4 seconds). This difference was statistically significant (F(1,169)=30.28, p<0.001), thus confirming our hypothesis.

Discussion

This task of solution validity assessment confirmed the effects observed on the first experiment, which involved a task of solution discovery. Even when participants were given the correct answer as well as the operation used to calculate it, they tended in more than 35% of the cases to reject it if the problem induced a representation that was semantically incongruent with it.

Similarly, even when participants managed to successfully identify the correct answers of cardinal problems, they needed more time than for ordinal problems, which, we believe, indicates semantic recoding.

These results suggest that the semantic congruence effects that have been previously highlighted are not restricted to the spontaneous access to a solving strategy, but also to the evaluation of a given solving strategy.

Conclusion

These two experiments confirmed that the process to overcome a semantic incongruence between an interpreted representation and a solving algorithm is a costly one. Both the success rates and the response times were influenced by the incongruence, therefore showing the prevalence of semantic aspects over mathematical ones when faced with a situation from which we do not spontaneously abstract a solving-relevant representation.

We believe that these results pave the way for the emergence of a new approach to problem solving, integrating the three main components of any problem solving activity: the procedural aspects (the algorithms), the representations inferred from the solver's knowledge about the world, and the mathematical structure of the problems. We think that characterizing the way these three levels of description interact would deeply contribute to the understanding of the underlying cognitive processes. In this study, we tried to describe how a representation inferred from the world semantics can prevent the solver from accessing the relevant solving algorithm. Even when this algorithm only consists in a mere subtraction.

This underlines the importance to conduct extensive work on mathematical reasoning from an educational perspective in order to help the learners overcome semantic congruence effects. The robustness of these processes among adults suggest that they cannot spontaneously disappear, and instead need to be the target of a substantial and focused teaching so as to be reduced. As it has already been highlighted (Bassok, Chase & Martin, 1998; Lee, DeWolf, Bassok, & Holyoak, 2016), mathematics textbooks tend to preferably use semantically congruent examples when teaching a specific arithmetic concept. Yet, our study seems to indicate that people face greater difficulties when trying to solve incongruent problems compared to congruent ones. Would it not be preferable, then, to teach students how to specifically address these situations, and how to resort to a semantic recoding of their initial representations? Maybe then, the remaining door would be available to them.

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